Recall

Def (Equidistribution)

Let $(x_n)_{n=1}^{\infty}$ be a sequence with $x_n \in [0,1)$.

$(x_n)_{n=1}^{\infty}$ is said to be equidistributed if

$\forall\ (a,b) \subset [0,1)$,

$$\lim_{N\to\infty} \frac{1}{N} \#\{ 1 \leq n \leq N;\ x_n \in (a,b) \} = b-a$$

Thm (Weyl's Criterion)

$(x_n)_{n=1}^{\infty}$ is equidistributed if and only if

$\forall\ k \in \mathbb{Z} \setminus \{0\}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k x_n} = 0$$

Sketch of proof

LHS $\lim \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int f(x)\,dx$ with $f = \chi_{(a,b)}$

RHS - . . . . all trigonometric polynomials.

($\Rightarrow$) We use step functions to approximate Riemann integrable functions.

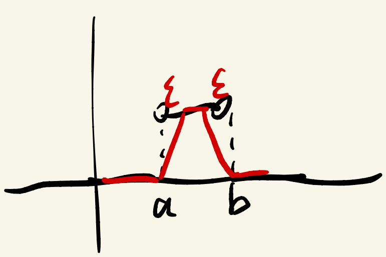
$\cup$

1-periodic continuous functions

$\cup$

trigonometric polynomials

($\Leftarrow$) We use trigonometric polynomials to approximate continuous then $\chi_{(a,b)}$



$\square$

(Weyl's Thm)

$(\{n\gamma\})_{n=1}^{\infty}$ is equidistributed if $\gamma \notin \mathbb{Q}$.

where $\{x\}$ is the fractional part of x.

- If $0<\sigma<1$ and $\gamma \neq 0$, then $(\{\gamma n^{\sigma}\})_{n=1}^{\infty}$ is equidistributed.

Pf: Fix any $k \in \mathbb{Z} \setminus \{0\}$. By Weyl's Criterion, it suffices to show

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k\gamma n^{\sigma}} \to 0 \quad \text{as } N \to \infty.$$

We write $a = k\gamma$. Then $a \neq 0$.

We want

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i a n^{\sigma}} \to 0 \quad \text{as } N \to \infty.$$

Step 1: $\left|\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i a n^{\sigma}} - \frac{1}{N}\int_1^{N+1} e^{2\pi i a x^{\sigma}}\,dx\right| \to 0$ as $N\to\infty$.

Step 2: $\left|\frac{1}{N}\int_1^{N+1} e^{2\pi i a x^{\sigma}}\,dx\right| \to 0$ as $N\to\infty$.

Step 1: $\left|\sum_{n=1}^{N} e^{2\pi i a n^{\sigma}} - \int_1^{N+1} e^{2\pi i a x^{\sigma}}\,dx\right|$

$$= \left|\sum_{n=1}^{N}\left(e^{2\pi i a n^{\sigma}} - \int_n^{n+1} e^{2\pi i a x^{\sigma}}\,dx\right)\right|$$

$$= \left|\sum_{n=1}^{N}\int_n^{n+1}\left(e^{2\pi i a n^{\sigma}} - e^{2\pi i a x^{\sigma}}\right)dx\right|$$

$$\leq \sum_{n=1}^{N}\int_n^{n+1}\left|e^{2\pi i a n^{\sigma}} - e^{2\pi i a x^{\sigma}}\right|dx$$

$$= \sum_{n=1}^{N}\int_n^{n+1}\left|e^{\pi i a(n^{\sigma}-x^{\sigma})} - e^{-\pi i a(n^{\sigma}-x^{\sigma})}\right|dx$$

$$= \sum_{n=1}^{N}\int_n^{n+1} 2\left|\sin a\pi(x^{\sigma}-n^{\sigma})\right|dx$$

$$\leq 2a\pi \sum_{n=1}^{N}\int_n^{n+1}(x^{\sigma}-n^{\sigma})\,dx$$

$|\sin x| \leq |x|$

$$\leq 2a\pi \sum_{n=1}^{N}\left[(n+1)^\sigma - n^\sigma\right]$$

$$= 2a\pi\left[(N+1)^\sigma - 1\right]$$

$$= o(N)$$ since $\sigma < 1$ ✓

step 2: $\left| \int_1^{N+1} e^{2\pi i a x^\sigma} dx \right|$

$$= \left| \int_1^{(N+1)^\sigma} e^{2\pi i a y} \frac{1}{\sigma} y^{\frac{1}{\sigma}-1} dy \right|$$

$y = x^\sigma$

$x = y^{\frac{1}{\sigma}}$

$\frac{dx}{dy} = \frac{1}{\sigma} y^{\frac{1}{\sigma}-1}$

$$= \left| \frac{1}{2\pi i a \sigma} \int_1^{(N+1)^\sigma} y^{\frac{1}{\sigma}-1} d\left(e^{2\pi i a y}\right) \right|$$

integrate by part

$$= \frac{1}{2\pi a \sigma} \left| y^{\frac{1}{\sigma}-1} e^{2\pi i a y} \Big|_{y=1}^{(N+1)^\sigma} - \int_1^{(N+1)^\sigma} \left(\frac{1}{\sigma}-1\right) y^{\frac{1}{\sigma}-2} e^{2\pi i a y} dy \right|$$

$$\left| y^{\frac{1}{\sigma}-1} e^{2\pi i a y} \Big|_{y=1}^{(N+1)^\sigma} \right| \le \left((N+1)^\sigma\right)^{\frac{1}{\sigma}-1} + 1$$

$$= (N+1)^{1-\frac{1}{\sigma}} + 1$$

$$= o(N)$$

$$\left| \int_1^{(N+1)^\sigma} \left(\frac{1}{\sigma}-1\right) y^{\frac{1}{\sigma}-2} e^{2\pi i a y} \, dy \right|$$

$$\le \int_1^{(N+1)^\sigma} \left(\frac{1}{\sigma}-1\right) y^{\frac{1}{\sigma}-2} \, dy$$

$$= \left(\frac{1}{\sigma}-1\right)^2 y^{\frac{1}{\sigma}-1} \Big|_1^{(N+1)^\sigma}$$

$$= \left(\frac{1}{\sigma}-1\right)^2 \left[(N+1)^{1-\frac{1}{\sigma}} - 1\right] = o(N).$$

$\square$

• $(\{\gamma \log n\})_{n=1}^{\infty}$ is not equidistributed for any γ.

Pf: By Weyl's Criterion, it suffices to find some $k \in \mathbb{Z} \setminus \{0\}$ s.t.

$$\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k\gamma \log n} \not\to 0 \quad \text{as } N \to \infty.$$

$$\text{"} \quad \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i \alpha \log n}$$

Step 1: $\left| \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i \alpha \log n} - \frac{1}{N}\int_1^{N+1} e^{2\pi i \alpha \log x}\, dx \right| \to 0$ as $N \to \infty$.

Step 2: $\left| \frac{1}{N}\int_1^{N+1} e^{2\pi i \alpha \log x}\, dx \right| \not\to 0$ as $N \to \infty$.

step 1:
$$\left| \sum_{n=1}^{N} e^{2\pi i a \log n} - \int_1^{N+1} e^{2\pi i a \log x} dx \right|$$

$$= \left| \sum_{n=1}^{N} \int_n^{n+1} \left(e^{2\pi i a \log n} - e^{2\pi i a \log x} \right) dx \right|$$

$$= \left| \sum_{n=1}^{N} \int_n^{n+1} \left(e^{\pi i a (\log n - \log x)} - e^{-\pi i a (\log n - \log x)} \right) dx \right|$$

$$\leq \sum_{n=1}^{N} \int_n^{n+1} \left| e^{\pi i a (\log n - \log x)} - e^{-\pi i a (\log n - \log x)} \right| dx$$

$$= \sum_{n=1}^{N} \int_n^{n+1} 2\pi a \left| \sin(\log n - \log x) \right| dx$$

$$\leq 2\pi a \sum_{n=1}^{N} \int_n^{n+1} (\log x - \log n) \, dx \qquad |\sin x| \leq |x|$$

$$\leq 2\pi a \sum_{n=1}^{N} \left[\log(n+1) - \log n \right]$$

$$= 2\pi a \log(N+1) = o(N)$$

Step 2:

$$\left|\frac{1}{N}\int_1^{N+1} e^{2\pi i a \log x}\,dx\right|$$

$$= \left|\frac{1}{N}\int_1^{N+1} x^{2\pi i a}\,dx\right|$$

$$= \left|\frac{1}{N}\cdot\frac{1}{2\pi i a+1}\,x^{2\pi i a+1}\Big|_{x=1}^{N+1}\right|$$

$$= \frac{(N+1)^{2\pi i a+1}}{N(2\pi i a+1)} - \underbrace{\frac{1}{N(2\pi i a+1)}}_{\to 0 \text{ as } N\to\infty}$$

Since $\left|\frac{(N+1)^{2\pi i a+1}}{N(2\pi i a+1)}\right| \to \left|\frac{1}{2\pi i a+1}\right| \neq 0$,

then $\frac{(N+1)^{2\pi i a+1}}{N(2\pi i a+1)} \not\to 0$ as $N\to\infty$.

$\square$

Rmk: • $(\{n\gamma\})_{n=1}^{\infty}$ is equidistributed, if $\gamma \notin \mathbb{Q}$

$(\{n^k\gamma\})_{n=1}^{\infty}$ is equidistributed

$\{n^k\gamma + n^{k-1}a_{k-1} + \cdots + a_0\}_{n=1}^{\infty}$ is equidistributed.